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BOUNDED 3-MANIFOLDS ADMIT NEGATIVELY CURVED METRICS WITH CONCAVE BOUNDARY

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Abstract

A metric can be constructed on any 3-manifold with nonempty boundary such that with respect to the metric the manifold has negative sectional curvature and the boundary is concave. In particular, the 3-ball admits such a metric.

Introduction

In this paper we construct a metric on any 3-manifold with boundary such that with respect to the metric the manifold has negative sectional curvature and the boundary is concave outwards. In particular, we construct such a metric on the 3-ball. This is surprising for several reasons.

Firstly, such a construction cannot be carried out in two dimensions. The Gauss-Bonnet theorem implies that the boundary of a negatively curved 2-disk is somewhere convex.

Secondly, such a metric cannot be constructed with constant negative sectional curvature. This contrasts with the recurrent theme in low-dimensional topology that negatively curved manifolds behave similarly to hyperbolic ones. Thus Thurston's geometrization conjecture states that a closed 3-manifold admitting a metric of negative sectional curvature also admits a hyperbolic metric, and Thurston has proved this for closed Haken manifolds and for bounded manifolds with totally geodesic boundary [4]. By contrast, there is no hyperbolic metric on the ball whose boundary is concave. Otherwise we could use the developing map [4] to immerse the ball into H^3 under a local isometry. An extremal point of the image would be a boundary point that could not be concave. Thus this construction can be viewed in some weak sense as giving negative evidence for the geometrization conjecture.

This metric has other strange properties. It is not induced by an immersion of the 3-ball into a complete negatively curved 3-manifold. The interior of the ball contains a null-homotopic closed geodesic. By contrast,

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in a complete negatively curved manifold there is a unique geodesic in each homotopy class.

The motivation for this construction was an attempt to generalize the 2π-theorem of Gromov and Thurston [1], which states that all but finitely many Dehn fillings on a cusped hyperbolic 3-manifold yield a closed, negatively curved 3-manifold. Cutting off a cusp of a hyperbolic 3-manifold along a horospherical torus gives a concave boundary to the hyperbolic manifold, and the metric can be extended over the solid torus added during Dehn filling so as to preserve negative curvature. The results of this paper show that the solid torus, like a hyperbolic manifold truncated along a cusp, admits a metric with negative sectional curvature and concave boundary. But all of the infinitely many Dehn fillings on the solid torus yield manifolds which do not admit a metric of negative sectional curvature. Thus the 2π-theorem does not generalize from the hyperbolic setting to the setting of variable negative curvature.

The paper is organized so as to first construct a metric on the 3-ball with negative sectional curvature and concave boundary, and then generalize the construction to obtain such metrics on arbitrary bounded 3-manifolds.

Construction of the metric on the 3-ball

The idea of the construction is to take a suitable hyperbolic manifold with totally geodesic genus-two boundary and to add two 2-handles to obtain a 3-ball. Pushing in the totally geodesic boundary slightly gives a concave boundary. The 2-handles can then be made to have a compatible negatively curved metric if the attaching curves are sufficiently long geodesics.

Lemma 1. *There is a 3-manifold X, obtained by removing an arc from $T^2 \times \{0\}$ to $T^2 \times \{1\}$ in $T^2 \times I$, which is hyperbolic and has totally geodesic boundary.*

Proof. An arc running from $T^2 \times \{0\}$ to $T^2 \times \{1\}$ whose complement is boundary incompressible, anannular, and atoroidal exists by the results of Meyers [2], [3]. Let X be the manifold obtained by removing a small open neighborhood of this arc. The double of X is a closed atoroidal 3-manifold, and admits a hyperbolic metric by Thurston's geometrization theorem for Haken manifolds. This manifold has an isometric involution preserving ∂X. It follows that ∂X is totally geodesic, proving the lemma.

In Example 1 we will give a more direct construction of a hyperbolic manifold with all the properties we need. This construction avoids the need to apply the less constructive results of [2], [3].

Let $\varepsilon > 0$ be a constant such that the boundary ∂X of X admits a collar neighborhood of width ε (ε can be chosen to depend only on the genus of ∂X, but for our purposes any $\varepsilon > 0$ will do). Let X_ε be the submanifold obtained from X by removing an ε-neighborhood of the boundary of X. Then ∂X_ε is a concave surface of constant normal curvature. Let $K(\varepsilon)$ be the normal curvature of ∂X_ε, so that $K(0) = 0$ and $K(\varepsilon) > 0$ for $\varepsilon > 0$. $K(\varepsilon)$ is also the normal curvature of the surface in H^3 which is at constant distance ε from a hyperbolic plane.

We now examine what happens when a 2-handle is added to X. Let m be a closed curve on ∂X, which is a meridian of the arc removed from $T^2 \times I$. ∂X is cut by m into two punctured tori, T_1 and T_2.

Lemma 2. *Any 2-handle addition to X along a non-separating curve in T_1 gives a solid torus. For any one of these, infinitely many distinct 2-handle additions along curves in T_2 give rise to a 3-ball.*

Proof. Compressing a punctured torus yields a 2-sphere. Thus any 2-handle addition along a non-separating curve in T_1 gives a manifold X' with a single torus as boundary. One way of constructing X' is to first take $T^2 \times I$, add a 2-handle to get a solid torus with a ball removed, and then remove an arc connecting the torus boundary component to the sphere boundary component. Such a construction always produces a solid torus.

Adding a second 2-handle to T_2 will in general give rise to a punctured lens space. However an infinite number of distinct 2-handles yield a ball, namely those where the attaching curve consists of one longitude and any number of meridians.

We now show that the 2-handle addition can be carried out with suitable control of the metrics. We construct a metric on the attaching 2-handles which is negatively curved, hyperbolic near the 2-handle boundary, and suitably concave. We begin by constructing a metric on the core of the 2-handle.

Lemma 3. *Given a positive constant $K(\varepsilon)$ there is a constant $L_\varepsilon > 0$ such that a disk D whose boundary has length$(\partial D) > L_\varepsilon$ admits a smooth metric h satisfying*:

1. *D is hyperbolic in a neighborhood of ∂D,*
2. *D is negatively curved everywhere,*
3. *∂D is convex with constant normal curvature $K(\varepsilon)$.*

Proof. We take polar coordinates (r, θ) on D and a metric of the form $h = dr^2 + f^2(r)\, d\theta^2$, where f is a function defined as follows. Fix $\varepsilon > 0$ and pick a constant $\alpha \in R$ and a smooth function $f(r)$ so that:

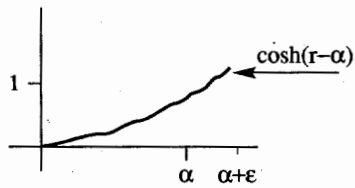

FIGURE 1

1. $f\colon [0, \alpha+\varepsilon] \to R$,
2. $f(0) = 0$,
3. $f(r) = c\sinh(r)$ for $r \in [0, \varepsilon]$,
4. $f(r) = \cosh(r-\alpha)$ for $r \in [\alpha+\varepsilon/2, \alpha+\varepsilon]$,
5. $f''(r) > 0$ for all r (i.e., f is convex).

The existence of such an f for large enough α follows immediately from Figure 1.

A calculation using Cartan's method of moving frames shows that the resulting curvature of the disk is given by $-f''/f$, $r > 0$. Explicitly, take orthonormal 1-forms

$$\omega^1 = dr, \qquad \omega^2 = f(r)\,d\theta.$$

Then

$$d\omega^1 = 0 = -\omega^1{}_2 \wedge \omega^2$$

and

$$d\omega^2 = f'\,dr \wedge d\theta = f'/f\omega^1 \wedge \omega^2 = -\omega^2{}_1 \wedge \omega^1.$$

So

$$\omega^1{}_2 = -f'/f\omega^2, \qquad d\omega^1{}_2 = -f''/f\omega^1 \wedge \omega^2.$$

The sectional curvature is given by $-f''/f$, so this metric on the disk has negative curvature, equal to -1 near $r = 0$ and near ∂D. The metric has a singularity at $r = 0$ with cone angle $2\pi c$. An appropriate branching gives a nonsingular metric on the disk with cone angle 2π. The boundary length then multiplies by $1/c$. Although the construction of f is restricted in that there is an upper bound to which value of c can be chosen, there is no lower bound, so that we can pick the cone angle to be anything less than some fixed constant c_0. As a result we can branch appropriately to get an arbitrary boundary length larger than some constant L_ε. The metric near a totally geodesic submanifold of a hyperbolic manifold has the form $dr^2 + \cosh^2 r\,dx^2$, where r measures the distance from the submanifold, and dx^2 is the hyperbolic metric on the submanifold (these are sometimes called Fermi coordinates). As a result the metric near the boundary of the

disk is the same as the metric on a hyperbolic surface at distances between $\varepsilon/2$ and ε from a geodesic.

We now extend this metric to a 3-dimensional 2-handle.

Lemma 4. *Given D as in Lemma 3, there is a metric on $D \times I$ such that*

1. *$D \times I$ is hyperbolic in a neighborhood of $\partial D \times I$,*
2. *$D \times I$ is negatively curved everywhere,*
3. *$\partial D \times I$ is convex with constant normal curvature $K(\varepsilon)$,*
4. *$D \times \partial I$ is concave.*

Proof. We first consider $D^2 \times R$ with coordinates (r, θ) on D^2 and ρ on R. Define the metric

$$g = d\rho^2 + \cosh^2 \rho\, h = d\rho^2 + \cosh^2 \rho\, dr^2 + \cosh^2 \rho\, f^2\, d\theta^2,$$

and calculate its curvature using moving frames.

Orthonormal 1-forms are

$$\omega^1 = d\rho, \qquad \omega^2 = \cosh \rho\, dr, \quad \omega^3 = f \cosh \rho\, d\theta$$

Differentiating gives

$$\begin{aligned}
d\omega^1 &= 0 = -\omega^1{}_2 \wedge \omega^2 - \omega^1{}_3 \wedge \omega^3, \\
d\omega^2 &= \sinh \rho\, d\rho \wedge dr = (\sin \rho / \cosh \rho)\omega^1 \wedge \omega^2 \\
&= -\omega^2{}_1 \wedge \omega^1 - \omega^2{}_3 \wedge \omega^3, \\
d\omega^3 &= f \sinh \rho\, d\rho \wedge d\theta + f \cosh \rho\, dr \wedge d\theta \\
&= \left(\frac{\sinh \rho}{\cosh \rho}\right) \omega^1 \wedge \omega^3 + \frac{f'}{f \cosh \rho} \omega^2 \wedge \omega^3 \\
&= -\omega^3{}_1 \wedge \omega^1 - \omega^3{}_2 \wedge \omega^2.
\end{aligned}$$

Set $\omega^i_j = a^i_{jk} \omega^k$. Then

$$\begin{aligned}
0 &= \omega^1{}_2 \wedge \omega^2 + \omega^1{}_3 \wedge \omega^3 \\
&= a^1{}_{21} \omega^1 \wedge \omega^2 + a^1{}_{23} \omega^3 \wedge \omega^2 + a^1{}_{31} \omega^1 \wedge \omega^3 + a^1{}_{32} \omega^2 \wedge \omega^3,
\end{aligned}$$

so $a^1{}_{21} = 0$, $a^1{}_{23} = a^1{}_{32}$, $a^1{}_{31} = 0$,

$$\begin{aligned}
\omega^1{}_2 &= a^1{}_{22} \omega^2 + a^1{}_{23} \omega^3, \\
\omega^1{}_3 &= a^1{}_{23} \omega^2 + a^1{}_{33} \omega^3, \\
\omega^2{}_3 &= a^2{}_{31} \omega^1 + a^2{}_{32} \omega^2 + a^2{}_{33} \omega^3;
\end{aligned}$$

$$(\sin\rho/\cosh\rho)\omega^1\wedge\omega^2 = -\omega^2{}_1\wedge\omega^1 - \omega^2{}_3\wedge\omega^3$$
$$= a^1{}_{22}\omega^2\wedge\omega^1 + a^1{}_{23}\omega^3\wedge\omega^1$$
$$- a^2{}_{31}\omega^1\wedge\omega^3 - a^2{}_{32}\omega^2\wedge\omega^3,$$

so $a^2{}_{32} = 0$, $a^1{}_{22} = -\sinh\rho/\cosh\rho$, $a^1{}_{23} = -a^2{}_{31}$,

$$\omega^1{}_2 = -(\sinh\rho/\cosh\rho)\omega^2 + a^1{}_{23}\omega^3,$$
$$\omega^1{}_3 = a^1{}_{23}\omega^2 + a^1{}_{33}\omega^3,$$
$$\omega^2{}_3 = -a^1{}_{23}\omega^1 + a^2{}_{33}\omega^3,$$

$$\left(\frac{\sinh\rho}{\cosh\rho}\right)\omega^1\wedge\omega^3 + \frac{f'}{f\cosh\rho}\,\omega^2\wedge\omega^3$$
$$= -\omega^3{}_1\wedge\omega^1 - \omega^3{}_2\wedge\omega^2 = a^1{}_{23}\omega^2\wedge\omega^1 + a^1{}_{33}\omega^3\wedge\omega^1$$
$$- a^1{}_{23}\omega^1\wedge\omega^2 + a^2{}_{33}\omega^3\wedge\omega^2,$$

so $a^1{}_{33} = -(\sinh\rho/\cosh\rho)$, $a^2{}_{33} = -f'/(f\cosh\rho)$, $a^1{}_{23} = 0$, and

$$\omega^1{}_2 = -(\sinh\rho/\cosh\rho)\omega^2,$$
$$\omega^1{}_3 = -(\sinh\rho/\cosh\rho)\omega^3,$$
$$\omega^2{}_3 = -(f'/f\cosh\rho)\omega^3.$$

We now calculate $\Omega = d\omega + \omega\wedge\omega$:

$$d\omega^1{}_2 = -d((\sinh\rho/\cosh\rho)\omega^2) = -d(\sinh\rho\,dr)$$
$$= -\cosh\rho\,d\rho\wedge dr = -\omega^1\wedge\omega^2,$$
$$d\omega^1{}_3 = -d((\sinh\rho/\cosh\rho)\omega^3) = -d(f(r)\sinh\rho\,d\theta)$$
$$= -f'(r)\sinh\rho\,dr\wedge d\theta - f(r)\cosh\rho\,d\rho\wedge\omega\,d\theta$$
$$= -\frac{f'(r)\sinh\rho}{f(r)\cosh^2\rho}\omega^2\wedge\omega^3 - \omega^1\wedge\omega^3,$$
$$d\omega^2{}_3 = d\left(\frac{f'}{f\cosh\rho}\omega^3\right) = -d(f'\,d\theta) = -f''\,dr\wedge d\theta$$
$$= -f''/(f\cosh^2\rho)\omega^2\wedge\omega^3,$$

$$(\omega \wedge \omega)^1{}_2 = \omega^1{}_3 \wedge \omega^3{}_2 = -(\sinh\rho/\cosh\rho)\omega^3 \wedge \frac{f'}{f\cosh\rho}\omega^3 = 0,$$

$$(\omega \wedge \omega)^1{}_3 = \omega^1{}_2 \wedge \omega^2{}_3 = -(\sinh\rho/\cosh\rho)\omega^2 \wedge -\frac{f'}{f\cosh\rho}\omega^3$$

$$= \frac{\sinh\rho f'}{f\cosh^2\rho}\omega^2 \wedge \omega^3,$$

$$(\omega \wedge \omega)^2{}_3 = \omega^2{}_1 \wedge \omega^1{}_3 = (\sinh\rho/\cosh\rho)\omega^2 \wedge -(\sinh\rho/\cosh\rho)\omega^3$$

$$= -(\sinh^2\rho/\cosh^2\rho)\omega^2 \wedge \omega^3,$$

$$\Omega^1{}_2 = -\omega^1 \wedge \omega^2,$$

$$\Omega^1{}_3 = -\frac{f'\sinh\rho}{f\cosh^2\rho}\omega^2 \wedge \omega^3 - \omega^1 \wedge \omega^3$$

$$+\frac{f'\sinh\rho}{f\cosh^2\rho}\omega^2 \wedge \omega^3$$

$$= -\omega^1 \wedge \omega^3,$$

$$\Omega^2{}_3 = -\left(\frac{f''}{f\cosh^2\rho} + \frac{\sinh^2\rho}{\cosh^2\rho}\right)\omega^2 \wedge \omega^3.$$

Now $\Omega^i{}_j = \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l$, so

$$R^1{}_{212} = -1, \quad R^1{}_{313} = -1, \quad R^2{}_{323} = -\frac{f''}{f\cosh^2\rho} + \frac{\sinh^2\rho}{\cosh^2\rho}.$$

Since f'' and f are positive, all the sectional curvatures $R^i{}_{jij}$ are negative. Moreover the metric is hyperbolic near $r = 0$ and near $\partial D \times R$.

We now construct a submanifold Y of $D \times R$ with the following properties:

1. Y is homeomorphic to $D \times [-1, 1]$.
2. $D \times \{-1\}$ and $D \times \{1\}$ are concave.
3. $\partial D \times [-1, 1]$ is convex with constant normal curvature $K(\varepsilon)$.
4. $D \times \{-1\}$ and $\partial D \times I$ are tangent at $\partial D \times \{-1\}$, and $D \times \{1\}$ and $\partial D \times I$ are tangent at $\partial D \times \{1\}$.

We first arrange for property 2. Note that S^1 acts isometrically on $D \times R$, via $\alpha \cdot (\rho, r, \theta) = (\rho, r, \alpha + \theta)$, $\alpha \in [0, 2\pi]$. We identify the orbit space O with the set of points $\theta = 0$. O sits inside Y as a totally geodesic submanifold, and inherits the hyperbolic metric $d\rho^2 + \cosh^2\rho\, dr^2$. Each circle orbit with $r > 0$ has a curvature vector which is invariant under the circle action. This gives a smooth equivariant vector

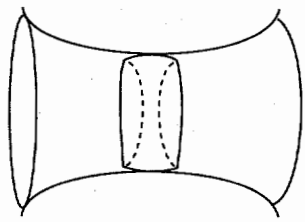

FIGURE 2A

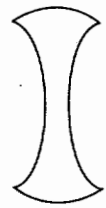

FIGURE 2B

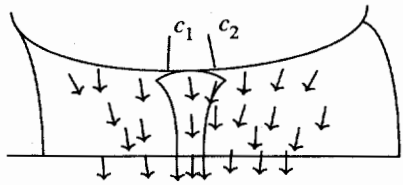

FIGURE 3

field on $\{D-(0,0)\}\times R$, which projects to a smooth vector field V on the interior of the orbit space, as depicted in Figure 3. At $\theta=0$, V is tangent to $D\times\{0)$ and points towards the center of this disk. As $r\to 0$, V becomes perpendicular to the $r=0$ axis. The direction of V, though not its magnitude, is illustrated in Figure 3.

Let c_1, c_2 be a pair of curves in O, each of which is convex outwards and transverse to V on $\text{int}(O)$, as in Figure 3. Construct surfaces of revolution C_1 and C_2 by taking the S^1 orbits of c_1 and c_2, and let W be the compact submanifold of $D\times R$ cut off by $C_1\cup C_2$. Since O is totally geodesic, c_1 and c_2 are principle curves of C_1 and C_2 respectively. The other principle curvature of C_1 and C_2 is the normal curvature of an S^1-orbit. This points outside W by construction, so that both principle curvatures are outward pointing and C_1 and C_2 are concave.

We now adjust W and C_1, C_2 near $\partial D\times R$ to obtain Y. W is hyperbolic near $\partial D\times R$, and $\partial D\times R$ has constant normal curvature $K(\varepsilon)$ in the θ direction, zero in the ρ direction. We construct the convex part of ∂Y by taking the submanifold of W with constant normal curvature $K(\varepsilon)$ in every direction on $\partial D\times I$. Finally we adjust C_1 and C_2 so that they meet the convex portion of ∂Y smoothly, as illustrated in Figure 2b.

Theorem 1. *Let M be a manifold which has negative curvature and concave boundary. Suppose that γ is a simple geodesic on the boundary ∂M of M, $\text{length}(\gamma) > L_\varepsilon$, and the metric on M in a neighborhood of γ is hyperbolic with constant normal curvature $K(\varepsilon)$. Then the manifold obtained by attaching a 2-handle along γ admits a negative curvature metric which is concave at the boundary.*

Proof. The geodesic γ has a δ-neighborhood for some $\delta > 0$. The 2-handle in Lemma 4 can be constructed so that its width is arbitrarily small, in particular less than δ. Attaching the 2-handle along γ gives the desired metric.

Theorem 2. *There exists a metric on a 3-dimensional ball B^3 which has negative sectional curvature and is concave at the boundary.*

Proof. We take the manifold $X(\varepsilon)$ constructed in Lemma 1 and attach a 2-handle to T_1 along a curve of length $> L_\varepsilon$. Theorem 1 implies that the resulting solid torus has a negative sectional curvature metric with concave boundary. In this metric the punctured torus T_2 in the boundary of the solid torus has a neighborhood which is hyperbolic, has constant normal curvature $K(\varepsilon)$, and has a geodesic boundary curve corresponding to the meridian m of the removed arc. It follows that there are geodesic curves in T_2 which have arbitrarily long length such that adding a 2-handle along any of these curves gives rise to a ball. Performing a 2-handle addition along one of these curves and applying Theorem 1 proves Theorem 2.

Geodesics in a complete negative sectional curvature manifold are always homotopically nontrivial. In contrast, for bounded negative sectional curvature manifolds we produce a closed null-homotopic geodesic contained in the interior of the manifold. This is not possible in dimension 2.

Corollary 1. *There exists a negative sectional curvature metric with concave boundary on the 3-ball in which the interior of the ball contains a null-homotopic closed geodesic.*

Proof. Take a closed geodesic in the hyperbolic structure on X which misses an ε-neighborhood of ∂X. The metric in a neighborhood of this curve is unchanged by the 2-handle additions.

Corollary 2. *There exists a negative sectional curvature metric on the 3-ball which is not induced by an immersion into any complete negatively sectional curved 3-manifold.*

Proof. If so, then the complete negatively sectional curved 3-manifold would contain a null-homotopic closed geodesic.

Construction of metrics on arbitrary bounded 3-manifolds

We next consider the question of what other manifolds admit negative sectional curvature metrics with concave boundary. A construction similar to the one of Lemma 1 shows that all 3-manifolds have such metrics.

Theorem 3. *There exists a metric on any orientable 3-manifold with nonempty boundary, which has negative sectional curvature and is concave at the boundary.*

Proof. Case 1. *M has a boundary component which is not a sphere.* Glue 3-balls to any spherical boundary components to get a manifold M', possibly coinciding with M. Remove an unknot K from a small ball in $\text{int}(M')$, the interior of M'. Using Myers' theorem, an arc α can be found running from K to a boundary component of M such that the complement X of an open neighborhood of $K \cup \alpha$ is boundary incompressible, atoroidal, and anannular [2], [3]. X admits a hyperbolic metric with ∂X consisting of horotori and totally geodesic surfaces. Push the totally geodesic surfaces in slightly to get concave boundary. Infinitely many distinct surgeries on K give rise to a manifold homeomorphic to M'. Choosing a sufficiently long surgery curve, as in Theorem 1, gives a metric with negative sectional curvature concave boundary on M'. Removing small balls if necessary, we obtain such a metric on M.

Case 2. *M has only 2-sphere boundary components.* Glue 3-balls to any spherical boundary components to get a closed manifold M'. Remove a two-component unlink from a small ball contained in M' (a Hopf link would also serve, as it did for the case of a 3-ball). Using Myers' theorem, remove an arc running from one of the components of this link to the other, so that the complement X of an open neighborhood of the resulting graph is boundary incompressible, atoroidal and anannular. This manifold X admits a hyperbolic metric with ∂X a totally geodesic, genus-2 surface. Push the boundary in slightly to get concave boundary. Just as before, infinitely many distinct pairs of Dehn surgeries on each component of the original unlink give rise to a manifold homeomorphic to $M' - \{B^3\}$. Choosing sufficiently long curves, as in Theorem 1, gives a metric with negative sectional curvature and concave boundary on $M' - \{B^3\}$. Removing more balls if necessary, we obtain a manifold homeomorphic to M.

Example 1. We give here a specific example of a manifold X which is hyperbolic, has totally geodesic genus-2 boundary, and such that on ∂X there are curves c_1 c_2 of arbitrarily long length with the property that adding 2-handles along c_1 and c_2 gives a 3-ball. Take X to be the complement of the handlebody in S^3 depicted in Figure 4. Thurston has observed that the complement of this handlebody can be decomposed into two truncated regular hyperbolic tetrahedra so as to have totally geodesic boundary [4]. By adding a 2-handle to a meridian of this handlebody, as indicated in Figure 4, one obtains a solid torus in S^3. Addition of another 2-handle then gives a 3-ball. Note that many more pairs of curves also yield B^3, as one can do "handle slides", i.e., given two curves, replace one of them by parallel copies of each connected by an arc in their complement.

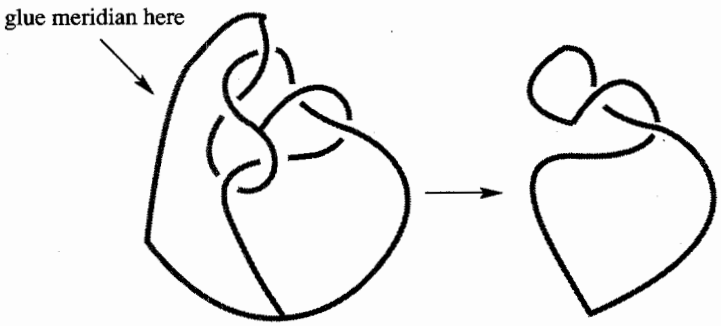

FIGURE 4

This process enables us to choose the attaching curves of the two handles arbitrarily long. Applying Theorem 1 gives the conclusion.

Question. Does every n-manifold with boundary admit a metric which has negative sectional curvature and is concave at the boundary, $n > 3$? In particular, does the n-ball?

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